A stochastic model for the semiclassical collective dynamics of charged beams in particle accelerators[1]

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Abstract

In this paper we briefly review the main aspects of a recent proposal to simulate semiclassical corrections to classical dynamics by suitable classical stochastic fluctuations, and we apply it to the specific instance of charged beams in particle accelerators. The resulting picture is that the collective beam dynamics, at the leading semiclassical order in Planck constant can be described by a particular diffusion process, the Nelson process, that is time–reversal invariant. Its diffusion coefficient $\sqrt{N}\lambda_c$ represents a semiclassical unit of emittance (here N is the number of particles in the beam, and λ_c is the Compton wavelength). The stochastic dynamics of the Nelson type can be easily recast in the form of a Schroedinger equation, with the semiclassical unit of emittance replacing Planck constant. Therefore we provide a physical foundation to the several quantum–like models of beam dynamics proposed in recent years. We also briefly touch upon applications of the Nelson and Schroedinger formalisms to incorporate the description of coherent collective effects.

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[1]To appear in the Proceedings of the International Workshop on "Quantum Aspects of Beam Dynamics", held in Stanford, 4–9 January 1998.

1 Introduction

The dynamical evolution of beams in particles accelerators is governed by external electromagnetic forces and by the interaction of the beam particles among themselves and with the environment. Charged beams are therefore higly nonlinear dynamical systems, and most of the studies on colliding beams rely either on classical phenomena such as nonlinear resonances, or they are concerned with isolated sources of unstable behaviors as building blocks of more complicated chaotic instabilities.

This line of inquiry has produced a general qualitative picture of dynamical processes in particle accelerators at the classical level. However, the coherent oscillations of the beam density and profile require, to be explained, some mechanism of local correlation and loss of statistical independence. This fundamental observation points towards the need to take into account all the interactions as a whole. Moreover, the overall interactions between charged particles and machine elements are really nonclassical in the sense that of the many sources of noise that are present, almost all are mediated by fundamental quantum processes of emission and absorbtion of photons. Therefore the equations describing these processes must be, in principle, quantum.

Starting from the above considerations, two different approaches to the classical collective dynamics of charged beams have been developed, one relying on the Fokker-Planck equation [1] for the beam density, another based on a mathematical coarse graining of Vlasov equation leading to a quantum–like Schroedinger equation, with a thermal unit of emittance playing the role of Planck constant [2].

The study of statistical effects on the dynamics of electron (positron) colliding beams by the Fokker–Planck equation has led to several interesting results, and has become an established reference in treating the sources of noise and dissipation in particle accelerators by standard classical probabilistic techniques [3].

Concerning the relevance of the quantum–like approach, at this stage we only want to point out that some recent experiments on confined classical systems subject to particular phase–space boundary conditions seem to to be well explained by a quantum–like (Schroedinger equation) formalism [4].

In any case, both approaches do not take into account quantum corrections, while in principle these effects should be relevant, expe-

cially in fixing fundamental lower limits to beam emittance. In this report we give a short summary of a recently proposed model for the description of collective beam dynamics in the semiclassical regime. This new approach relies on the idea of simulating semiclassical corrections to classical dynamics by suitable classical stochastic fluctuations with long range coherent correlations, whose scale is ruled by Planck constant [5].

The fluctuative hypothesis has been introduced by simple stability criteria, and it has been semiquantitatively tested for many stable systems, including beams. The virtue of the proposed semiclassical model is twofold: on the one hand it can be formulated both in a probabilistic (Fokker–Planck) fashion and in a quantum–like (Schroedinger) setting. It thus bridges the formal gap between the two approaches. At the same time it goes further by describing collective effects beyond the classical regime due to the semiclassical quantum corrections.

In particular, implementing the fluctuative hypothesis qualitatively by simple dimensional analysis, we derive a formula for the phase–space unit of emittance that connects in a nontrivial way the number of particles in the beam with Plank constant.

The fluctuative scheme is then implemented quantitatively by introducing a random kinematics in the form of a diffusion process in configuration space for a generic representative of the beam (collective degree of freedom).

We are interested in the description of the stability regime, when thermal dissipative effects are balanced on average by the RF energy pumping, and the overall dynamics is conservative and time–reversal invariant in the mean. Therefore, we model the random kinematics with a particular class of diffusion processes, the Nelson diffusions, that are nondissipative and time–reversal invariant (We will briefly comment at the end of the last section on the extension of the present scheme to include the treatment of dissipative effects).

The diffusion process describes the effective motion at the mesoscopic level (interplay of thermal equilibrium, classical mechanical stability, and fundamental quantum noise) and therefore the diffusion coefficient is set to be the semiclassical unit of emittance provided by qualitative dimensional analysis. In other words, we simulate the quantum corrections to classical deterministic motion (at leading order in Planck constant) with a suitably defined random kinematics replacing the classical deterministic trajectories.

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Finally, the dynamical equations are derived via the variational principle of classical dynamics, with the only crucial difference that the kinematical rules and the dynamic quantities, such as the Action and the Lagrangian, are now random. The stochastic variational principle leads to a pair of coupled equations for the beam density and the beam center current velocity, describing the dynamics of beam density oscillations. It is an effective description in the stability regime.

The stochastic variational principle for Nelson diffusions (with diffusion coefficient equal to Planck constant) is a well developed mathematical tool that has originally been introduced to provide a stochastic formulation of quantum mechanics. Therefore, apart from the different objects involved (beam spatial density versus Born probability density; Planck constant versus emittance), the dynamical equations of our model formally reproduce the equations of the Madelung fluid (hydrodynamic) representation of quantum mechanics. In this sense, the present scheme allows for a quantum–like formulation equivalent to the probabilistic one.

At the end of the last section we will briefly discuss how the hydrodynamic formulation of the equations for the collective stochastic dynamics can be used to control the beams, for instance by selecting the form of the external potential needed to obtain coherent oscillations of the beam density.

2   Simulation of semiclassical effects by classical fluctuations

Let us consider a physical system subject to a classical force law of modulus $F(r)$ that is attractive and confining at least for some finite space region with a characteristic linear dimension R. Given N elementary granular constituents of the system, each of mass m, let v denote their characteristic velocity, and τ their characteristic unit of time.

A characteristic unit of action per particle is then defined as

$$\alpha = mv^2\tau. \tag{1}$$

If the system has to be stable and confined, one must impose that the characteristic potential energy of each particle be on average equal

to its characteristic kinetic energy (virial theorem):

$$\mathcal{L} \cong mv^2, \tag{2}$$

where $\mathcal{L}$ is the work performed by the system on a single constituent. On the other hand, if the system extends on the characteristic length scale R,

$$\mathcal{L} \cong NF(R)R. \tag{3}$$

By equations (2) and (3) we can express the characteristic velocity v as

$$v \cong \sqrt{\frac{NF(R)R}{m}}. \tag{4}$$

Introducing the global time scale $\mathcal{T}$ associated to the system, we also have $v = R/\mathcal{T}$. Replacing this expression and equation (4) for each power of v in equation (1), we obtain the following expression for the action per particle:

$$\alpha \cong \sqrt{mF(R)}R^{3/2}N^{1/2}\frac{\tau}{\mathcal{T}}. \tag{5}$$

Mechanical stability requires that the action per particle must not depend on N, while on the other hand, the microscopic unit of time τ must obviously depend on N and on the system's global time scale $\mathcal{T}$. Therefore we must impose

$$\tau = \frac{\mathcal{T}}{\sqrt{N}}. \tag{6}$$

Inserting equation (6) into equation (5) we obtain the unit of action per particle as a explicit expression in terms of the constituent's mass, the system's linear dimension R, and the classical force calculated in R:

$$\alpha \cong m^{1/2}R^{3/2}\sqrt{F(R)}. \tag{7}$$

The scaling relation (6) can be also interpreted as a fluctuative hypothesis connecting the time scale of a microscopic stochastic motion with the classical time scale of the global system. In fact, equation (6) was first postulated by F. Calogero in his attempt to prove that quantum mechanics might be interpreted as a tiny chaotic component of the individual particles' motion in a gravitationally interacting Universe [6].

In our scheme, rather than being a postulated consequence of classical gravitational chaos, the fluctuative hipothesis of Calogero derives from a condition of mechanical stability. Since the stability conditions and the virial theorem apply to any classically stable and confined system, even with a small number of degrees of freedom, our derivation of equations (6) and (7) is universal as it applies to any interactions, not only gravity, and to systems composed by any number of constituents, not necessarily large, and not necessarily classically chaotic.

We have verified that for any stable aggregate, plugging in equation (7) the pertaining interaction F, individual constituents' mass m and aggregate's linear dimension R, one has that *the unit of action per particle α is always equal, in order of magnitude, to Planck action constant h.*

Our interpretation of this remarkable result is then that the fluctuative relation (6) and the associated formula (7) for the Planck quantum of action simulate (reformulate) in a classical probabilistic language the Bohr-Sommerfeld quantization condition. They provide a classical description of quantum corrections to classical phase–space dynamics at the leading semiclassical order h.

We here briefly derive the result for the case of interest of a stable bunch of charged particles in a particle accelerator. We consider a single electron (proton), in the reference frame comoving with the bunch. Confinement and stability of the bunch arise from the many complicated interactions among its constituents and between the same constituents and the external magnetic and RF fields. The net effect can be, in first approximation, schematized by saying that the single electron (proton) experiences an effective harmonic force, the typical phenomenological law of force for beams when higher anharmonic contributions can be neglected: $F(R) \cong KR$, where K is the effective phenomenological elastic constant. We then have for beams:

$$\alpha \cong m^{1/2} R^2 K^{1/2}. \tag{8}$$

Let us consider for instance the transverse oscillations for protons at Hera: in this case we have $K = 10^{-12} N m^{-1}$, the linear transverse dimension of the bunch $R = 10^{-7} m$, and the proton mass. For electrons in linear colliders we have instead $K = 10^{-11} N m^{-1}$, $R = 10^{-7} m$, and the electron mass. In both cases, from equation (8) we have that the unit of action per particle α, ruling the coherence and stability of the bunch, is in both cases h, up to at most one order of magnitude.

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All other instances of charged bunches considered lead to the same result, yielding our first important conclusion: the stability of charged beams is ruled by quantum effects on a mesoscopic scale. Moreover, at the semiclassical level, such quantum aspects can be described in terms of suitable classical fluctuations that mimick (simulate) the weak but unavoidable presence of fundamental quantum noise.

The parameter that rules the stability of the system at the mesoscopic scale is however not directly h, but in the case of charged beams some characteristic unit of emittance. This is a scale of action, or of length when divided by the Compton wavelength, that measures the spread of the bunch in phase space, or, equivalently, in real space.

This notion is very useful in the regime of stability and of thermal equilibrium that we explicitly consider. In this case the emittance can be expressed as a unit of equivalent thermal action. To introduce a characteristic unit of emittance in our fluctuative semiclassical scheme we then proceed as follows: the time scale of quantum fluctuations is defined as the ratio between h and a suitable energy describing the equilibrium state of the given system. This leads naturally to identify this energy with the equivalent thermal energy $k_B T$, with k_B the Boltzmann constant and T the equivalent temperature. On the other hand, in our scheme such time scale coincides with the fluctuative time τ; we therefore have:

$$\tau \cong \frac{h}{k_B T} \,. \tag{9}$$

Using relation (6) we obtain the equivalent thermal unit of action

$$k_B T \mathcal{T} \cong h\sqrt{N} \,. \tag{10}$$

Introducing the Compton wavelength $\lambda_c = h/mc$ and dividing by it both sides of equation (10) we finally obtain the characteristic unit of emittance $\mathcal{E}$:

$$\mathcal{E} \cong \lambda_c \sqrt{N} \,. \tag{11}$$

Equation (11) connects in a nontrivial way the number of particles in a given charged beam and the Compton wavelength. The square root of N appears as a semiclassical "memory" of quantum interference. The relation (11) seems to point out the existence of a mesoscopic lower bound on the emittance some orders of magnitude above the quantum limit given by the Compton wavelength. Moreover, Equation (11) yields the correct order of magnitude in for the

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emittance in typical accelerators: for instance, with $N \cong 10^{11} \div 10^{-12}$, one has $\mathcal{E} \cong 10^{-6}m$ in excellent agreement with the lowest emittances that are at the moment experimentally attainable.

Actually, limits and requirements on beam existence, luminosity and statistics do not allow for beams with a number of particles appreciably lower than $N \cong 10^{10} \div 10^{11}$. Thus the estimate (11) really provides an *a priori* lower bound, as it implies that the emittance cannot be reduced appreciably below the mesoscopic thresholds $\mathcal{E} \cong 10^5 \div 10^6 \lambda_c$, well above the Compton wavelength limit and only one or two orders of magnitude below the current experimental limits. It seems also unlikely that further quantum corrections beyond the leading semiclassical order could somehow contribute in lowering the mesoscopic bound (11) as a function of N.

3   Stochastic collective dynamics in the stability regime

The previous discussion can be made more quantitative by observing that the fluctuative relation (6) can be be recast with a little bit of work in the alternative form

$$l \sim \tau^{2/3} \,, \tag{12}$$

where l is a characteristic mean free path per particle. The detailed derivation of relation (12) from equation (6) is reported elsewhere [5]. Relation (12) indicates that the classical flcutuative simulation of semiclassical corrections really implies a fractal space–time relation in the mean, with a Kepler exponent associated to stable, confined and coherent dynamical systems, for instance charged beams in the stability regime.

We therefore model the spatially coherent fluctuations (6) and (12) by a random kinematics performed by some collective degree of freedom $q(t)$ representative of the beam. The most universal continuous random kinematics that we can choose is a diffusion process in real or configuration space. In this way the random kinematics provides an effective description of the space–time variations of the particle beam density $\rho(x,t)$ as it coincides with the probability density of the diffusion process performed by $q(t)$.

Since it measures a collective effect at the mesoscopic scale, the diffusion coefficient must be related to the equilibrium parameter in the stability regime, that is to the characteristic semiclassical unit of emittance (11) rather than to the Plank action constant.

Then, in suitable units, the basic stochastic kinematical relation is the Ito stochastic differential equation

$$dq(t) = v_+(q,t)dt + \mathcal{E}^{1/2}dw, \tag{13}$$

where v_+ is the deterministic drift, the square root of the characteristic emittance (11) is the diffusion coefficient, and dw is the time increment of the standard δ–correlated Wiener noise.

We are concerned with the regime of stability of the beam oscillation dynamics, both since it is the relevant regime in the physics of accelerators and because the beam can be considered quasistationary during it, until, eventually, space charge effects become dominant and the beam is lost. In such stationary regime the energy lost by photonic emissions is regained in the RF cavities, and on average the dynamics is still time–reversal invariant. We can therefore still define a classical Lagrangian $L(q, \dot{q})$ for the system, however with the classical deterministic kinematics replaced by the random diffusive kinematics (13).

The equations of dynamics can then be deduced from the classical Lagrangian by simply modifying the variational principles of classical mechanics into stochastic variational principles. In fact, the mathematical techniques of stochastic variational principles have been developed and applied to obtain Nelson stochastic mechanics, an independet stochastic reformulation of quantum mechanics in terms of time–reversal invariant Markov diffusion processes with diffusion coefficient $\hbar$ [7]. In the context of Nelson stochastic mechanics one derives Schroedinger equation in the form of the Madelung coupled hydrodynamic equations for the probability density and the probability current [7].

In the present mesoscopic context the analysis is quite similar to that of Nelson stochastic mechanics, yielding again two coupled nonlinear hydrodynamic equations, however, with the emittance (11) replacing Planck constant in the diffusion coefficient, the real space bunch density replacing the quantum mechanical probability density, and the bunch center velocity replacing the quantum mechanical probability current.

We now briefly sketch the derivation of the dynamical equations. The detailed analysis may be found elsewhere [8]. Given the stochastic differential equation (13) for the diffusion process $q(t)$ in $d = 3$ space dimensions, one introduces the classical Lagrangian

$$L(q, \dot{q}) = 1/2m\dot{q}^2 - V(q) \,. \tag{14}$$

For the generic trial diffusion $q(t)$ one has, respectively, the probability density $\rho(x, t)$, the forward drift $v_+(x, t)$ and the backward drift $v_-(x, t)$. It is then useful to define two new variables, $v(x, t)$ and $u(x, t)$, respectively the current velocity and the osmotic velocity, defined as:

$$v = \frac{v_+ + v_-}{2} \;; \quad u = \frac{v_+ - v_-}{2} = \mathcal{E}\frac{\nabla\rho}{\rho} \,. \tag{15}$$

The mean classical action is defined in strict analogy to the classical action in the deterministic case, but for the limiting procedure that needs to be taken in the sense of expectation values, as the sample paths of a diffusion process are non differentiable:

$$A(t_0, t_1; q) = \int_{t_0}^{t_1} lim_{\Delta t \to 0^+} E\left[\frac{m}{2}\left(\frac{\Delta q}{\Delta t}\right)^2 - V(q)\right] dt \,, \tag{16}$$

where E denotes the expectation with respect to the probability density ρ. It can be shown that the mean classical action (16) associated to the diffusive kinematics (13) can be cast in the following particularly appealing Eulerian hydrodynamic form [7]:

$$A(t_0, t_1; q) = \int_{t_0}^{t_1} dt \int d^3x \left[\frac{m}{2}\left(v^2 - u^2\right) - V(x)\right] \rho(x, t) \,. \tag{17}$$

The stochastic variational principle now follows: the Action is stationary, $\delta A = 0$, under smooth variations of the density $\delta\rho$, and of the current velocity δv, with vanishing boundary conditions at the initial and final times, if and only if the current velocity is the gradient of some scalar field $S(x, t)$ (the phase):

$$mv = \nabla S \,. \tag{18}$$

With the above conditions met, the two coupled nonlinear Lagrangian equations of motion for the density ρ (or alternatively for

the osmotic velocity u) and for the current velocity v (or alternatively for the phase S) are the Hamilton–Jacobi–Madelung equation:

$$\partial_t S + \frac{m}{2}v^2 - 2m\mathcal{E}^2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + V(x) = 0\,, \qquad (19)$$

and the continuity equation:

$$\partial_t \rho = -\nabla[\rho v]. \qquad (20)$$

By solving equations (19) and (20) the state of the bunch is completely determined. Formal linearization of the equations can be achieved through the standard De Broglie ansatz yielding the Schroedinger equation of the quantum–like models. However, one should bear in mind that the real hydrodynamic equations (19)–(20) are the physically fundamental objects, while linearizing them to a complex Schroedinger equation is a bare mathematical tool that can be useful for calculational needs, but bears no physical significance. In particular, the complex wave function is devoid of any physical meaning. Thus, in the present context, the situation is just the opposite to that in quantum mechanics, where instead the wave function and the Schroedinger equation are the fundamental physical ingredients.

The observable structure is quite clear: the expectations (first moments) of the three components of the current velocity v are the average velocities of oscillation of the bunch center along the longitudinal and transverse directions. The expectations (first moments) of the three components of the process $q(t)$ give the average coordinate of the bunch center. The second moments of $q(t)$ allow to determine the dispersion (spreading) of the bunch. In the harmonic case, these are all the moments that are needed (Gaussian probability density), and we have coherent state solutions. In the anharmonic case the coupled equations of dynamics may be used to achieve a controlled coherence: given a desired state (ρ, v) the equations of motion (19) and (20) can be solved for the external controlling potential $V(x, t)$ that realizes the desired state. Lack of space prevents us from commenting further on this very important application of our formalism. A thorough and detailed study of the controlled coherent evolutions in the framework of our stochastic model will be presented in a forthcoming paper [9].

4   Aknowledgement

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